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An Approach to Wegner's Estimate Using Subharmonicity

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Abstract We develop a strategy to establish a Wegner estimate and localization in random lattice Schrödinger operators on $\mathbb{Z}^d$, which does not rely on the usual eigenvalue variation argument. Our assumption is that the potential $V(\omega)$ depends real analytically on ω and we use a distributional property of analytic functions in many variables. An application is given to models where V_n is a self-adjoint matrix obtained by random unitary conjugation $V_n = U_n A U_n^*$ of a fixed matrix A.

Keywords Random Schrödinger operators · Localization

0 Introduction

We consider lattice Schrödinger operators $H = \Delta + V$ on $\mathbb{Z}^d$, with Δ the lattice Laplacian and V a potential (typical classes are random or quasi-periodic potentials). We are only discussing the localized regime (point spectrum with Anderson localization) and $d > 1$ (if $d = 1$, other techniques such as the transfer matrix formalism become available). Considering quasi-periodic models (cf. [2, 3]), the difficulty with classical eigenvalue methods (as applied in [5] for instance) are high multiplicities or near-multiplicities that seem to make those problems intractable with such techniques. An alternative approach (for real analytic potentials), in some sense more robust, was gotten from the theory of subharmonicity applied to suitable determinants (see [1] for instance). This technology bypasses the eigenvalue multiplicity problems and gives a quick general approach to estimating Green's functions by multi-scale analysis. The applications so far involved mainly quasi-periodic models and variants, where $V_n = V_{n_1,\dots,n_d} = V(T_1^{n_1} \cdots T_d^{n_d} x)$ and $T_1, \dots, T_d$ are shifts or skew-shifts acting on a d-torus $\mathbb{T}^d$ (note that also the theory of semi-algebraic sets played an essential role in this analysis). If on the other hand one considers random potentials $V_n = V(\omega_{n_1}, \dots, \omega_{n_d})$,

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the usual procedure to bound the Green's function at a fixed energy E is a simple first order eigenvalue variation (Wegner's estimate). The motivation for this Note is a certain class of random Hamiltonians (see Sect. 3) where eigenvalue methods turn out to be again problematic.

This lead us to rework the quasi-periodic technology in the random case (considering real-analytic potentials) to obtain a theory of same generality. Compared with the quasi-periodic case, and, more generally, deterministic dynamics, the random case is of course easier since at different sites the potential takes independent values; there is in particular no need for semi-algebraic set theory. On the other hand, as our functions depend now on a large number of variables, an essentially dimension free version of the so-called 'Cartan Lemma' on the level sets of an analytic function is desirable.

An appropriate statement (not fully dimension free) is proven in Sect. 1 of the paper. Note that in fact completely dimension free results of this type are true and appear in [6] (their setting as stated, is not quite adequate for our application).

Next one repeats the arguments from [1, Chap. 14], to obtain a suitable matrix-valued Cartan-type theorem that applies in the setting of random *SO*. See Sect. 2 of the paper, where we give an outline without all technical details.

In Sect. 3, we return to the original motivation. It is a model proposed by J. Fröhlich, where the random-potential V is given by $V_n = U_n^* A U_n$, with $A = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix}$ and U_n a random element in $SU(2)$.

We prove Anderson localization at the edge of the spectrum.

1 A Matrix-Valued Cartan-Type Theorem

The generalization of Proposition 4.1 from [1] to the setting of real analytic random potentials relies on the following:

Lemma 1 *Let F be a real analytic function on $\Omega = [-\frac{1}{2}, \frac{1}{2}]^n$ which extends to an analytic function on D^n, $D = \{z \in \mathbb{C}; |z| < 1\}$, with a bound*

$$|F(z_1, \ldots, z_n)| \le 1. \tag{1.1}$$

Assume further there is $a \in \Omega$ such that

$$|F(a)| > \varepsilon \tag{1.2}$$

where $0 < \varepsilon < \frac{1}{2}$. Denoting for $\delta > 0$

$$E_\delta = \{x \in \Omega \mid |F(x)| < \delta\} \tag{1.3}$$

we have

$$|E_\delta| < Cn\delta^{\frac{c}{\log 1/\varepsilon}} \tag{1.4}$$

where $0 < c, C$ are constants

(we also use $|\ |$ to denote "measure" or "cardinality").

Proof The claim is derived from the classical statement for $n = 1$. Using polar coordinates (setting the origin at a), write

$$|E_\delta| = c_n \int_{S_{n-1}} \int_0^{r(\zeta)} 1_{E_\delta}(a + r\zeta) r^{n-1} dr d\zeta \tag{1.5}$$

where $\Omega = \{a + r\zeta \,|\, \zeta \in S_{n-1} \text{ and } 0 \le r \le r(\zeta)\}$.

For fixed $\zeta \in S_{n-1}$, $f(r) = F(a + r\zeta)$ is a real analytic function of $r \in I = [0, r(\zeta)]$. It extends to an analytic function $\tilde{f}(z)$ on the neighborhood $\mathcal{D} = I + \frac{1}{2}\max(1, r(\zeta)).D \subset \mathbb{C}$ of I, where $\max_{z\in\mathcal{D}} |\tilde{f}(z)| \le 1$ and $|\tilde{f}(0)| > \varepsilon$. From Cartan's lemma, it follow that

$$\int 1_{E_\delta}(a + r\zeta) dr = |\{r \in I \,|\, |f(r)| < \delta\}| < C\delta^{\frac{c}{\log 1/\varepsilon}} r(\zeta) \tag{1.6}$$

with c, C constants. Substituting (1.6) in (1.5) gives the bound

$$c_n \left(\int_{S_{n-1}} r(\zeta)^n d\zeta \right) \delta^{\frac{c}{\log \frac{1}{\varepsilon}}} < n\, \delta^{\frac{c}{\log \frac{1}{\varepsilon}}} \tag{1.7}$$

in (1.4). □

Remark The important point in (1.4) is a reasonable dependence on the dimension n. In fact, [6] contains a fully dimension-free result, but formulated with Euclidean balls in $\mathbb{C}^n$ rather than polydisics (the polydisic formulation is the setting needed here).

Next we prove an analogue of [1, Proposition 14.1], with an n-dimensional parameter.

Lemma 2 *Let $A(x)$ be a real analytic self-adjoint $N \times N$ matrix function of $x \in [-\frac{1}{2}, \frac{1}{2}]^n$, satisfying the following conditions* (*with $M \ll N$ and $B_1, B_2, B_3 > 1$*).

(1.7) *$A(x)$ has an analytic extension $A(z)$ to $z \in D^n$ with*

$$\|A(z)\| < B_1 \quad \text{for } z \in D^n \tag{1.8}$$

(1.9) *There is a subset Λ of $[1, N]$ such that*

$$|\Lambda| \le M \tag{1.10}$$

and for all $z \in D^n$

$$\|(R_{[1,N]\backslash\Lambda} A(z) R_{[1,N]\backslash\Lambda})^{-1}\| < B_2 \tag{1.11}$$

(*here R_S denotes coordinate restriction to S*).

(1.12) $\|A(a)^{-1}\| < B_3$ *for some* $a \in [-\frac{1}{2}, \frac{1}{2}]^n$.

Then

(1.13) $\text{mes}\{x \in [-\frac{1}{2}, \frac{1}{2}]^n \,|\, \|A(x)^{-1}\| > e^t\} < Cne^{-\frac{ct}{M \log B_1 B_2 B_3}}$.

Proof The main idea is to reduce the inversion of $A(x)$ to that of a small matrix. Consider the following analytic matrix-valued function on D^n (with index set Λ)

$$B(z) = R_\Lambda A(z) R_\Lambda - (R_\Lambda A(z) R_{\Lambda^c})(R_{\Lambda^c} A(z) R_{\Lambda^c})^{-1}(R_{\Lambda^c} A(z) R_\Lambda) \tag{1.14}$$

satisfying by (1.8), (1.11)

$$\|B(z)\| < 2B_1^2B_2 \quad \text{for } z \in D^n. \tag{1.15}$$

The invertibility of $A(x)$ is equivalent to that of $B(x)$ and more precisely

$$\begin{aligned}\|B(x)^{-1}\| &\lesssim \|A(x)^{-1}\| \lesssim (1+\|(R_{\Lambda^c}A(x)R_{\Lambda^c})^{-1}\|^2)(1+\|B(x)^{-1}\|)\\ &\lesssim B_2^2(1+\|B(x)^{-1}\|).\end{aligned} \tag{1.16}$$

We argue as follows. Since $B(x)$ is selfadjoint and (1.16), (1.12)

$$|\det B(a)| = \prod_{\lambda\in \operatorname{Spec} B(a)} |\lambda| \geq \|B(a)^{-1}\|^{-M} > (cB_3)^{-M}. \tag{1.17}$$

Also, by Cramer's rule and (1.15)

$$\|B(x)^{-1}\| \leq \frac{\|B(x)\|^M}{|\det B(x)|} < \frac{(2B_1^2B_2)^M}{|\det B(x)|}. \tag{1.18}$$

Consider the analytic function on D^n

$$F(z) = (2B_1^2B_2)^{-M}\det B(z). \tag{1.19}$$

Hence $|F(z)| \leq 1$ by (1.15) and $|F(a)| > (CB_1^2B_2B_3)^{-M}$ by (1.17).

Application of Lemma 1 with $\varepsilon = (CB_1^2B_2B_3)^{-M}$ to F implies

$$\begin{aligned}\operatorname{mes}\left[x\in\left[-\frac{1}{2},\frac{1}{2}\right]^n \,\Big|\, |\det B(x)|<\delta\right] &< Cn\left(\frac{\delta}{(2B_1^2B_2)^M}\right)^{\frac{c}{M\log CB_1^2B_2B_3}}\\ &< Cn\delta^{\frac{c}{M\log B_1B_2B_3}}.\end{aligned} \tag{1.20}$$

If $|\det B(x)| > \delta$, then $\|A(x)^{-1}\| < (2B_1^2)^M B_2^{M+2}\delta^{-1}$ by (1.16), (1.18), so that (1.13) follows from (1.20). □

2 Application to Random Schrödinger Operators

Let $d, r \in \mathbb{Z}_+$ and Δ the lattice Laplacian on $\mathbb{Z}^d$. Let

$$V(\omega) = \sum_{\delta\in\mathbb{Z}^d} v(\omega_j)e_j \otimes e_j \tag{2.0}$$

with $v(x)$ a real-analytic function on $[-\frac{1}{2},\frac{1}{2}]^\nu$ ranging in the self-adjoint $r\times r$ complex matrices; the ω_j are independent variables. Let

$$H(\omega) = V(\omega) + \Delta$$

be the lattice Schrödinger operator on $\mathbb{Z}^d$ acting on $\mathbb{C}^r$-functions.

Take $\nu = 1$ for notational simplicity and assume $v(x)$ extends to a bounded analytic function D.

Fix an energy E and denote $Q_N = [1, N]^d$, $R_N = R_{Q_N}$,

$$H_N(\omega) = R_N H(\omega) R_N$$

and

$$G_N(E;\omega) = \big(H_N(\omega) - E + io\big)^{-1}.$$

Rather than Wegner's estimate, we consider the following statement involving also off-diagonal decay.

$$(*) \begin{cases} \text{There is a subset } \Omega_N \subset [-\tfrac{1}{2}, \tfrac{1}{2}]^{Q_N} \text{ such that} \\ \qquad |\Omega_N| < N^{-K} & (2.1) \\ \text{and for } \omega \in [-\tfrac{1}{2}, \tfrac{1}{2}]^{Q_N} \backslash \Omega_N \\ \qquad \|G_N(E;\omega)\| < e^{N^{1/2}} & (2.2) \\ \text{and} \\ \qquad |G_N(E;\omega)(j,j')| < e^{-cN} \quad \text{for } |j-j'| > \dfrac{N}{10}. & (2.3) \end{cases}$$

Here K is a sufficiently large power (depending on d).

Assuming $(*)$ valid at some initial large scale N_0, our aim is to establish inductively this property at all scales.

It turns out that for $d \geq 2$, besides cubes, we need to consider the larger class of so-called "fundamental regions" $\mathcal{E}_N$ (= differences of N-cubes) in order to be able to exploit the resolvent identity (due to corners).

A detailed exposition of these higher dimensional aspects of the multi-scale analysis may be found in [3] and we will largely ignore them here.

Assume $(*)$ holds at scales $N \leq N_1$. Hence, if $Q \in \mathcal{E}_N$, $N \leq N_1$ and $G_Q(E;\omega) = (R_Q H(\omega) R_Q - E + io)^{-1}$, there is an exceptional set $\Omega_Q \subset [-\frac{1}{2}, \frac{1}{2}]^Q$, $|\Omega_Q| < N^{-K}$, such that $G_Q(E;\omega)$ satisfies (2.2), (2.3) for $\omega \notin \Omega_Q$.

Let $N_2 < N_1^2$ and $N_3 = N_2^{\frac{1}{8d}} < N_1^{\frac{1}{4}}$. Denote $B = Q_{N_2}$.

Let $K_1 = 100d$.

Considering N_3-regions $Q = j + Q_{N_3} \subset B$, $Q_{N_3} \in \mathcal{E}_{N_3}$, it follows from $(*)$ at scale N_3 that the set $\Omega^{(1)} \subset [-\frac{1}{2}, \frac{1}{2}]^B$ for which there are at least K_1 disjoint N_3-regions $Q \subset B$ such that each $G_Q(E;\omega)$ fails (2.2) + (2.3), has measure at most

$$N_2^{dK_1} N_3^{-KK_1} < N_3^{-\frac{1}{2}KK_1} \tag{2.4}$$

provided we let

$$K \geq 10d^2. \tag{2.5}$$

Fix $\omega \notin \Omega^{(1)}$. From (2.2), (2.3) and a standard application of the resolvent identity (requiring also property (2.3)), we construct a decomposition $B = \Lambda \cup (B\backslash\Lambda)$ with the following properties.

(2.6) Λ is a union of at most K_1 cubes Q of size $N_3 \leq N(Q) < 20K_1^2 N_3$ and so that

$$\operatorname{dist}(Q, Q') > 10K_1N_3 \quad \text{for distinct } Q, Q'.$$

(2.7) $\|G_{B\backslash\Lambda}(E;\omega)\| \lesssim e^{N_3^{1/2}}$.
(2.8) $|G_{B\backslash\Lambda}(E;\omega)(j,j')| < e^{-c|j-j'|}$ for $|j-j'| > \frac{N_3}{10}$.

Moreover (2.7), (2.8) are also valid for any $Q \in \mathcal{E}_N$, $N \geq N_3$, $Q \subset B\backslash\Lambda$.

Next, we make the following further construction. Let Q be one of the components of Λ and let

$$Q \subset Q_{(1)} \subset Q_{(2)} \subset \cdots \subset Q_{(K_1)}$$

with $Q_{(i+1)}$ an N_3-neighborhood of $Q_{(i)}$. Since $|\Omega_{Q_{(i)}}| < N_3^{-K}$, the set

$$\pi_i = \mathrm{Proj}_{[-\frac{1}{2},\frac{1}{2}]^{Q_{(i)}\backslash Q_{(i-1)}}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^{Q_{(i)}}\Big\backslash \Omega_{Q_{(i)}}\right) \subset \left[-\frac{1}{2},\frac{1}{2}\right]^{Q_{(i)}\backslash Q_{(i-1)}}$$

certainly satisfies

$$|\pi_i| > 1 - N_3^{-K}. \tag{2.9}$$

Returning to $\omega \in [-\frac{1}{2},\frac{1}{2}]^B$, we also exclude the event that for some cube $Q \subset B$, $N_3 \leq N(Q) < 20K_1^2 N_3$ and $Q \subset Q_{(1)} \subset \cdots \subset Q_{(K_1)} \subset B$ as above, we have

$$(\omega_j)_{j\in Q_{(i)}\backslash Q_{(i-1)}} \notin \pi_i \quad \text{for all } 1 \leq i \leq K_1. \tag{2.10}$$

By (2.9), this condition removes a further subset $\Omega^{(2)} \subset [-\frac{1}{2},\frac{1}{2}]^B$ of measure

$$|\Omega^{(2)}| < N_2^{d+1}(N_3)^{-KK_1} < N_3^{-\frac{1}{2}KK_1}. \tag{2.11}$$

Let $\omega \notin \Omega^{(1)} \cup \Omega^{(2)}$. Then for each component Q of Λ, there is some $1 \leq i \leq K_1$ such that $(\omega_j)_{j\in Q_{(i)}\backslash Q_{(i-1)}} \notin \pi_i$ and we replace Q by this $Q_{(i-1)}$. Note that by (2.6), the new cubes are still $10N_3$-separated. We then redefine Λ as the union of these larger cubes, so that Λ contains the original region and (2.7), (2.8) still hold. Moreover Λ has the additional property that each of its component Q has an N_3-neighborhood Q' such that

$$(\omega_j)_{j\in Q'\backslash Q} \in \mathrm{Proj}_{[-\frac{1}{2},\frac{1}{2}]^{Q'\backslash Q}}\left(\left[-\frac{1}{2},\frac{1}{2}\right]^{Q'}\backslash\Omega_{Q'}\right). \tag{2.12}$$

Hence, for each Q', there is $\omega^{(Q')} \in [-\frac{1}{2},\frac{1}{2}]^{Q'}\backslash\Omega_{Q'}$ with $\omega_j = \omega_j^{(Q')}$ for $j \in Q'\backslash Q$. Define the element $\omega' \in [-\frac{1}{2},\frac{1}{2}]^B$ as follows

$$\begin{cases} \omega'_j = \omega_j & \text{if } j \in B\backslash\Lambda, \\ \omega'_j = \omega_j^{(Q')} & \text{if } j \in Q. \end{cases} \tag{2.13}$$

Thus $G_{B\backslash\Lambda}(E;\omega') = G_{B\backslash\Lambda}(E;\omega)$ satisfies (2.7), (2.8) and each component Q of Λ has an N_3-neighborhood Q' such that $\|G_{Q'}(E;\omega')\| = \|G_{Q'}(E;\omega^{(Q')}\| < e^{10K_1N_3^{1/2}}$.

Also $\mathrm{dist}(Q'_1, Q'_2) > 10N_3$ for distinct components Q_1, Q_2.

The usual F-S coupling lemma then implies that

$$\|G_B(E;\omega')\| < e^{10K_1N_3^{1/2}}. \tag{2.14}$$

Fix the variables $(\omega_j)_{j\in B\backslash\Lambda}$ and apply Lemma 2 to the real-analytic matrix function on $[-\frac{1}{2},\frac{1}{2}]^\Lambda$

$$A(x_j, j\in\Lambda) = H_B(E;\omega_j(j\in B\backslash\Lambda), x_j(j\in\Lambda)) - E \tag{2.15}$$

(replacing N by $|B| = N_2^d$ and $n = M = |\Lambda| < K_1(20K_1^2N_3 + K_1N_3)^d < (30K_1^2N_3)^d$ in Lemma 2).

In (1.7), $B_1 = B_1(v)$. In (1.9), clearly $R_{B\backslash\Lambda}A(z)R_{B\backslash\Lambda} = G_{B\backslash\Lambda}(E;\omega)$ does not depend on $z\in D^\Lambda$ and, by (2.7) (1.11) holds with $B_2 = e^{N_3^{1/2}}$.

Finally, condition (1.12) is satisfied for $a = (\omega_j')_{j\in\Lambda}\in[-\frac{1}{2},\frac{1}{2}]^\Lambda$ and $B_3 = e^{10K_1N_3^{1/2}}$, by (2.13), (2.14). It follows from (1.13) that

$$\begin{aligned}&\text{mes}\left\{x\in\left[-\frac{1}{2},\frac{1}{2}\right]^\Lambda \,\middle|\, \|G_B(E;\omega_j(j\in B\backslash\Lambda), x_j(j\in\Lambda))\| > e^t\right\}\\ &\quad< |\Lambda| e^{-\frac{ct}{|\Lambda|.K_1.N_3^{1/2}}} < (30K_1^2N_3)^d\, e^{-(K_1^2N_3)^{-d-1}t}.\end{aligned} \tag{2.16}$$

Note that the number of possible regions Λ is certainly bounded by $N_2^{(d+1)K_1}$. It follows that

$$\|G_B(E;\omega)\| < e^{N_2^{\frac{1}{2}}}. \tag{2.17}$$

for $\omega\notin\Omega_{N_2}$, where by (2.4), (2.11), (2.16)

$$\begin{aligned}|\Omega_{N_2}| &\le |\Omega^{(1)}| + |\Omega^{(2)}| + N_2^{(d+1)K_1} \quad (2.16)\\ &< N_3^{-\frac{1}{2}KK_1} + N_2^{2dK_1}e^{-(K_1^2N_3)^{-d-1}N_2^{\frac{1}{2}}}\\ &< N_2^{-2K}\end{aligned} \tag{2.18}$$

(by the choice of N_3 and K_1).

To conclude the induction, it remains to verify the off-diagonal decay (2.3) at scale N_2. Let $\Lambda\subset B = Q_{N_2}$ be the set introduced above, which is a union of at most K_1 cubes of size $\sim N_3$. Let $N_2' = \frac{N_2}{100K_1}$. We may construct a cover of a N_2'-neighborhood of Λ by cubes Q of size $N_2'\le N(Q) < 2K_1N_2'$ and at mutual distance at least N_2'. By preceding Green's function bound at scale $\le N_2$, we may ensure that

$$\|G_Q(E;\omega)\| < e^{N_2^{1/2}} \tag{2.19}$$

for any N_2'-cube $Q\subset Q_{N_2}$, provided we exclude an ω-set of measure at most

$$N_2^{d+1}(N_2')^{-2K} < N_2^{-\frac{3}{2}K} \tag{2.20}$$

(invoking (2.18)). Equation (2.3) then follows from (2.7), (2.8), (2.19) and another application of the coupling lemma.

This completes the inductive step. Recalling (2.5), we may take $K = 10d^2$.

Remarks

(1) We may of course more generally assume $v(x)$ a real-analytic selfadjoint matrix function defined on an open subset of a real analytic variety in $\mathbb{R}^\nu$.

(2) Instead of the lattice Laplacian Δ, we may take any operator $T(i, j) = \tau(i - j)$, $i, j \in \mathbb{Z}^d$, where $\tau(k) = \tau(-k)$ is a self-adjoint $\mathbb{C}^r$-matrix, $\|\tau(k)\|$ decaying exponentially in $|k|$.

With $(*)$ at our disposal, the usual technology to prove Anderson localization applies (see for instance [4] for a treatment involving a very weak assumption on the measure $|\Omega_N|$ of the exceptional set in (2.1), and which certainly covers the present situation).

Hence we may state the following

Theorem *Let $H(\omega)$ be as above and let $I \subset \mathbb{R}$ be an interval. Assume there is a sufficiently large N_0 such that for all $E \in I$*

$$\|G_{N_0}(E; \omega)\| < e^{N_0^{1/2}} \tag{2.21}$$

and

$$|G_{N_0}(E; \omega)(j, j')| < e^{cN_0} \quad \textit{for } |j - j'| > \frac{N_0}{10} \tag{2.22}$$

for ω taken outside a set of measure at most $N_0^{-20d^2}$ ("sufficiently large" depends on the potential function $v(x)$ and the constant c in (2.22)).

Then, almost surely in ω, $H(\omega)$ can only have point spectrum with exponentially decaying eigenfunctions for energies $E \in I$.

Remark A typical setting where the assumption of the theorem is easy to fulfill is by restricting the energy range to a neighborhood of the edge of the spectrum (see next section for an application).

3 An Application

The following model was our original motivation.

Let A be a self-adjoint $\mathbb{C}^r$-matrix with eigenvalues $\lambda_1 > \lambda_2 \geq \cdots \geq \lambda_r$ and define $v(U) = U^* A U$ with U ranging in $G = SU(r)$ for instance. Hence $v(U)$ is a real-analytic self-adjoint matrix function on G.

Consider the Hamiltonian

$$H(\omega) = V(\omega) + \Delta$$

on $\mathbb{Z}^d$, with $V(\omega)$ defined by (2.0) and $\omega \in G^{\mathbb{Z}^d}$.

The upper spectral edge of H is

$$E_* = \lambda_1 + 2d. \tag{3.1}$$

Note that if one attempts the usual approach to a Wegner estimate by eigenvalue variation $\frac{\partial\lambda(\omega)}{\partial\omega_i}$, $H(\omega)\xi(\omega) = \lambda(\omega)\xi(\omega)$, we get $\frac{\partial\lambda(\omega)}{\partial\omega_j}\big|_{\omega_j=I} = 0$ if $A\xi_j \wedge \xi_j = 0$. This explains the difficulty with the eigenvalue perturbation argument. On the other hand, previous Theorem applies for E near E_*. We verify the initial scale assumption.

Let $N = N_0$. Estimating

$$G_N(E; \omega) = E^{-1}(1 - E^{-1}H_N(\omega))^{-1} \tag{3.2}$$

with a Neumann series, one has to analyze what it means for ω that

$$\|H_N(\omega)\| > E_* - \kappa. \tag{3.3}$$

If (3.3), there is $\xi \in \ell^2_{Q_N}(\mathbb{C}^r)$, $|\xi| = 1$ such that

$$\sum_{j\in Q_N} \langle AU_j\xi_j, U_j\xi_j\rangle + \sum_{\substack{j,j'\in Q_N\\ |j-j'|=1}} \langle \xi_j, \xi_{j'}\rangle = \langle H_N(\omega)\xi, \xi\rangle > E_* - \kappa. \tag{3.4}$$

Hence, by (3.1)

$$\sum \langle AU_j\xi_j, U_j\xi_j\rangle > \lambda_1 - \kappa \tag{3.5}$$

and

$$\sum_{|j-j'|=1} \langle \xi_j, \xi_{j'}\rangle > 2d - \kappa \tag{3.6}$$

(set $\xi_j = 0$ for $j \notin Q_N$).

Denote $\eta \in \mathbb{C}^r$, $|\eta| = 1$ the eigenvector of A with eigenvalue λ_1. From (3.5)

$$1 - \frac{\kappa}{\lambda_1 - \lambda_2} < \sum |\langle U_j\xi_j, \eta\rangle|^2 \le \sum Re\langle \xi_j, e^{i\varphi_j}|\xi_j|U_j^*\eta\rangle \tag{3.7}$$

for some $\varphi_j \in [0, 2\pi[$. Therefore

$$\sum |\xi_j - e^{i\varphi_j}|\xi_j|U_j^*\eta|^2 < \frac{2\kappa}{\lambda_1 - \lambda_2}. \tag{3.8}$$

Denote $e_1, \dots, e_d$ the unit vector basis of $\mathbb{Z}^d$. From (3.6) we get for $\alpha = 1, \dots, d$

$$\sum \langle \xi_j, \xi_{j+e_\alpha}\rangle > 1 - \kappa$$

implying

$$\sum |\xi_j - \xi_{j+e_\alpha}|^2 < 2\kappa \tag{3.9}$$

and also

$$\Big(\sum |\xi_j - \xi_{j+k}|^2\Big)^{\frac{1}{2}} < 2d\ell\sqrt{\kappa} \tag{3.10}$$

for $k = (k_1, \dots, k_d) \in \mathbb{Z}^d$, $|k_\alpha| \le \ell$.

From (3.8), (3.9), $\sum |\xi_j|^2 \, |e^{i\varphi_{j+e_\alpha}} U^*_{j+e_\alpha}\eta - e^{i\varphi_j}U_j^*\eta|^2 < C\kappa$ and hence

$$\sum |\xi_j|^2 \, |\langle U_{j+e_\alpha}U_j^*\eta, \eta\rangle|^2 > 1 - c\kappa. \tag{3.11}$$

Fix some $\alpha = 1, \dots, d$. It follows from (3.10), (3.11) that for $k \in \mathbb{Z}^d$ as above

$$\sum |\xi_j|^2 |\langle U_{j+2k+e_\alpha}U^*_{j+2k}\eta, \eta\rangle|^2 = \sum |\xi_{j-2k}|^2 |\langle U_{j+e_\alpha}U_j^*\eta, \eta\rangle|^2 > 1 - c\ell\sqrt{\kappa}. \tag{3.12}$$

Let $\ell \sim \kappa^{-1/3}$ and average (3.12) over $k \in \{0, 1, \dots, \ell - 1\}^d$. We obtain some $j \in Q_N$ such that

$$\sum_{0\le k_\alpha<\ell} |\langle U_{j+2k+e_\alpha}U^*_{j+2k}\eta, \eta\rangle|^2 > \frac{3}{4}\ell^d. \tag{3.13}$$

Since the $U_{j+2k+e_\alpha} U^*_{j+2k}$ are independent random variables in G and $\int_G |\langle U\eta, \eta\rangle|^2 dU = \frac{1}{r}$, (3.13) holds with probability at most $\exp(-c\ell^d)$. Taking $\kappa \sim (\log N)^{3/d}$, it follows that (3.3) fails outside a set Ω_N with $|\Omega_N| < N^{-20d^2}$. Taking $N = N_0$ large enough, the assumptions of the Theorem are verified with $I = [E_* - \frac{1}{(\log N_0)^4}, E_*]$ and $c \sim (\log N_0)^{-3/d}$.

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